

FANO SCHEMES OF LINES ON TORIC SURFACES

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ABSTRACT. We completely describe the Fano scheme of lines $\mathbf{F}_1(X)$ for a projective toric surface X in terms of the geometry of the corresponding lattice polygon.

1. INTRODUCTION

Let X be a projective variety embedded in \mathbb{P}^n . The Fano scheme of lines $\mathbf{F}_1(X)$ is the fine moduli space parametrizing lines of \mathbb{P}^n contained in X . Such Fano schemes have been studied extensively in the case of hypersurfaces (e.g. [BVdV79] [HMP98] [Beh06]), beginning with the classical theorem of Cayley-Salmon that a non-singular cubic surface contains exactly 27 lines.

In this short note, we completely describe the Fano scheme of lines $\mathbf{F}_1(X)$ for a projective toric surface X . More specifically, let $X \subset \mathbb{P}^n$ be a normal toric surface embedded in \mathbb{P}^n by the complete linear system of a very ample divisor D . The data of X and D corresponds to a lattice polygon $P \subset \mathbb{R}^2$, see e.g. [Ful93]. In our main result (Theorem 2.2), we completely describe $\mathbf{F}_1(X)$ in terms of the geometry of the polygon P .

In §2 we introduce necessary background and state our main result, which we then prove in §3. As an easy consequence, we deduce that if X is a non-singular toric surface, then $\mathbf{F}_1(X)$ is also non-singular (Corollary 2.4). We conclude in §4 by considering examples as well as applications of our result to non-toric surfaces. Indeed, our result coupled with degeneration techniques may be used to give upper bounds on the number of lines on certain non-toric surfaces.

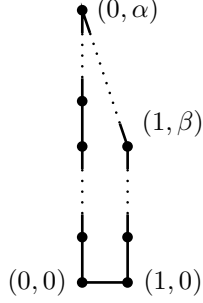
2. PRELIMINARIES AND MAIN RESULT

2.1. Fano schemes. Let K be an algebraically closed field and $X \subset \mathbb{P}_K^n$ any projective K -scheme. For any natural number $k \in \mathbb{N}$, the projective scheme $\mathbf{F}_k(X)$ is the fine moduli space parametrizing k -planes of \mathbb{P}^n contained in X , see e.g. [EH00, §IV.3] for a detailed description. If T is an algebraic torus acting on \mathbb{P}^n which fixes X , then the Fano schemes $\mathbf{F}_k(X)$ inherit a natural T -action.

2.2. Projective toric varieties. Consider an m -dimensional polytope $P \subset \mathbb{R}^m$ whose vertices are lattice points in \mathbb{Z}^m . Set

$$X_P = \text{Proj } K[S_P]$$

where S_P is the subsemigroup of $\mathbb{Z}^m \times \mathbb{Z}$ generated by $(P \cap \mathbb{Z}^m) \times 1$. Then X_P is an m -dimensional projective toric variety (with action by the torus $T = \text{Spec } K[\mathbb{Z}^m]$) and any normal projective toric variety embedded by a complete linear system is of this form, see e.g. [Ful93, §3.4]. In the case of primary interest to us ($m = 2$), the

FIGURE 1. The polytope $P_{\alpha, \beta}$ for a rational normal scroll

converse is true, that is, X_P is always normal and embedded by a complete linear system.

The torus fixed points of the Fano schemes $\mathbf{F}_k(X_P)$ are easy to describe. By a *primitive k -simplex* we mean the convex hull of $k+1$ lattice points in \mathbb{Z}^m containing no other lattice points.

Proposition 2.1. *The torus fixed points of $\mathbf{F}_k(X_P)$ are in bijection with the set of faces of P which are primitive k -simplices.*

Proof. Any fixed point of $\mathbf{F}_k(X_P)$ must be the closure of a k -dimensional T -orbit in X_P , which are in bijection to the set of k -dimensional faces of P [Ful93, §3]. Given a face $Q \subset P$, the corresponding orbit closure as a subvariety of \mathbb{P}^n is simply X_Q sitting inside of a linear subspace of \mathbb{P}^n . But X_Q is a linear space if and only if there are no relations among the lattice points of Q , that is, Q is a primitive k -simplex. \square

2.3. Rational normal scrolls. We say that a toric surface X_P is a *rational normal scroll* if P is lattice equivalent to a polyhedron of the form

$$P_{\alpha, \beta} = \text{conv}\{(0, 0), (1, 0), (0, \alpha), (1, \beta)\} \quad \alpha, \beta \in \mathbb{Z}_{\geq 0} \quad a > 0 \quad \alpha \geq \beta,$$

see Figure 1. Note that $X_{P_{\alpha, \beta}}$ is embedded in $\mathbb{P}^{\alpha+\beta+1}$ with coordinates x_u , $u \in \mathbb{Z}^2 \cap P$ and cut out by the 2×2 minors of the matrix

$$(1) \quad \begin{pmatrix} x_{(0,0)} & x_{(0,1)} & \cdots & x_{(0,\alpha-1)} & x_{(1,0)} & x_{(1,1)} & \cdots & x_{(1,\beta-1)} \\ x_{(0,1)} & x_{(0,2)} & \cdots & x_{(0,\alpha)} & x_{(1,1)} & x_{(1,2)} & \cdots & x_{(1,\beta)} \end{pmatrix}.$$

Note that as an abstract variety, $X_{P_{\alpha,0}}$ is the (weighted) projective space $\mathbb{P}(1, 1, \alpha)$, and for $\beta \geq 1$, $X_{P_{\alpha, \beta}}$ is the Hirzebruch surface

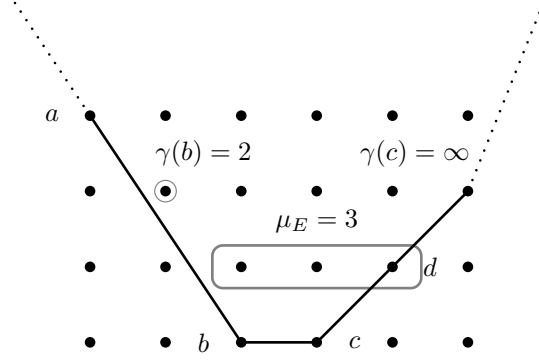
$$\mathcal{F}_{\alpha-\beta} = \text{Proj}_{\mathbb{P}^1} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha - \beta)).$$

2.4. Lattice geometry. Let $v, w \in \mathbb{Z}^2$ be primitive vectors. Then there exists a unique $p, q \in \mathbb{Z}$, $0 \leq q < p$ with q, p relatively prime such that $A(v) = (1, 0)$ and $A(w) = (-q, p)$ for some $A \in GL_2(\mathbb{Z})$ [Ful93, §2.2]. Concretely, $p = |\det(v, w)|$ and if α, β are integers such that $\alpha v_1 + \beta v_2 = 1$, then q is the p less the remainder of $\alpha w_1 + \beta w_2$ by p . We define

$$\gamma(v, w) := \lceil p/q \rceil.$$

Note that if $q = 0$, we adopt the convention that $\gamma(v, w) = \infty$.

Now let P be a lattice polygon in \mathbb{R}^2 . By Proposition 2.1, we see that fixed points of $\mathbf{F}_1(X_P)$ correspond to primitive edges of P . For each primitive edge $E = \overline{bc}$ of

FIGURE 2. The quantities μ_E , $\gamma(b)$, and $\gamma(c)$

P , we will define invariants γ_b , γ_c , and μ_E . Indeed, let a, d be the other lattice points on the boundary of P adjacent to b and c respectively. Then

$$\begin{aligned}\gamma_b &:= \gamma(c - b, a - b); \\ \gamma_c &:= \gamma(b - c, d - c).\end{aligned}$$

Likewise, let $u_E \in (\mathbb{Z}^2)^*$ be the primitive functional such that $u_E(b) = u_E(c)$, and $u_E(v) \geq u_E(b)$ for all $v \in P$. We define

$$\mu_E := \#\{v \in P \cap \mathbb{Z}^2 \mid u_E(v) = u_E(b) + 1\}.$$

Note that $\mu_E \geq 1$.

These quantities have easy geometric interpretations, see Figure 2 for an illustration. The quantity μ_E is the number of lattice points of P in “height one” above the edge E . On the other hand, $\gamma(b)$ is the height in which a lattice point of P first appears “to the left” of the ray extending from b through the left-most lattice point in height one. Likewise, $\gamma(c)$ is the height in which a lattice point of P first appears “to the right” of the ray extending from b through the right-most lattice point in height one.

2.5. Main result. We now are able to formulate our main result. For any $l \in \mathbb{N}$, set

$$\xi_l := \operatorname{Spec} K[z]/z^l.$$

Theorem 2.2. *Let X_P be a projective normal toric surface.*

(1) *If $P = P_{\alpha, \beta}$, then*

$$\mathbf{F}_1(X_P) = \begin{cases} \mathbb{P}^2 & \alpha = 1, \beta = 0 \\ \mathbb{P}^1 \sqcup \mathbb{P}^1 & \alpha = \beta = 1 \\ \mathbb{P}^1 \times \xi_2 & \alpha \geq 2, \beta = 0 \\ \mathbb{P}^1 \sqcup \xi_1 & \alpha \geq 2, \beta = 1 \\ \mathbb{P}^1 & \alpha \geq 2, \beta \geq 2 \end{cases}$$

(2) *If X is not a rational normal scroll, then $\mathbf{F}_1(X_P)$ is either zero-dimensional or empty, with irreducible components in bijection to the primitive edges E*

of P . For each such edge $E = \overline{bc}$, the corresponding irreducible component Z_E has the form

$$Z_E = \begin{cases} \xi_1 & \mu_E \geq 2 \\ \xi_{\min\{\gamma(b), \gamma(c)\}} & \mu_E = 2 \\ \xi_{\gamma(b)} \times \xi_{\gamma(c)} & \mu_E = 1 \end{cases}.$$

In particular, we have

$$\deg Z_E = \begin{cases} 1 & \mu_E \geq 2 \\ \min\{\gamma(b), \gamma(c)\} & \mu_E = 2 \\ \gamma(b) \cdot \gamma(c) & \mu_E = 1 \end{cases}.$$

Remark 2.3. Suppose that X_P is not a rational normal scroll. If $\mu_E = 2$, then either $\gamma(b)$ or $\gamma(c)$ must be finite. Likewise, if $\mu_E = 1$, then both $\gamma(b)$ and $\gamma(c)$ are finite. Hence, the quantities appearing in part 2 of the theorem are finite.

Corollary 2.4. *Let X_P be a non-singular projective toric surface. Then $\mathbf{F}_1(X_P)$ is non-singular. In particular, it is reduced.*

Proof. Recall that X_P is non-singular if and only if for every vertex v of P , the primitive lattice vectors in the directions of the outgoing edges form a lattice basis [Ful93, §2.1, 3.4]. It is straightforward to check that if there is some edge E of P such that $\mu_E \leq 2$, then X_P must be a rational normal scroll. Furthermore, the case $P = P_{\alpha, \beta}$ for $\alpha \geq 2, \beta = 0$ cannot occur, since it is not smooth. The claim now follows directly from Theorem 2.2. \square

3. PROOF OF MAIN RESULT

Throughout this section, we fix a lattice polygon P and consider the toric surface X_P . To each lattice point $u \in \mathbb{Z}^2 \cap P$ associate a variable x_u ; we consider the \mathbb{Z}^2 -graded ring

$$R = K[x_u \mid u \in \mathbb{Z}^2 \cap P]$$

with grading given by $\deg x_u = u$. Then X_P is embedded in $\text{Proj } R$; its ideal I_P is generated by binomials corresponding to homogeneous relations among the lattice points of P , and is homogeneous with respect to the \mathbb{Z}^2 -grading of R .

We will begin the proof by describing the Fano scheme in an affine neighborhood of a given toric fixed point of $\mathbf{F}_1(X_P)$. By Proposition 2.1, such a fixed point corresponds to a primitive edge $E = \overline{bc}$ of P , which we now fix. It will be useful to apply a lattice transformation to bring P into a standard form. Let a and d be the other lattice points on the boundary of P adjacent to b and c , respectively. After applying an invertible affine lattice transformation to \mathbb{Z}^2 , we may assume that $b = (0, 0)$, $c = (1, 0)$, $a = (-q, p)$ for $p > 0$, $q \geq 0$, $p > q$ and q, p relatively prime, arriving at something similar to what is pictured in Figure 2. Note that now μ_E is simply equal to

$$\mu_E = 1 + \max\{\lambda \in \mathbb{Z} \mid (\lambda, 1) \in P\}$$

and $\gamma(b) = \lceil p/q \rceil$.

The line L_E on X_P corresponding to E is given by the vanishing of the coordinates x_u , where $u = (u_1, u_2)$ satisfies $u \in P \cap \mathbb{Z}^2$ and $u_2 > 0$. We recall the

description of $\mathbf{F}_1(X_P)$ in a neighborhood of this line, see e.g. [EH00, §IV.3]. Let S be the \mathbb{Z} -graded ring

$$S = K[s, t, \sigma_u, \tau_u \mid u \in P \cap \mathbb{Z}^2, u \neq b, c]$$

with grading given by $\deg s = \deg t = 0$, $\deg \sigma_u = \deg \tau_u = u_2$ for $u = (u_1, u_2)$. We may view the coordinates σ_u, τ_u as parametrizing an open subset of lines in $\text{Proj } R$ by considering the line between the points $(x_b = 1, x_c = 0, x_u = \sigma_u)$ and $(x_b = 0, x_c = 1, x_u = \tau_u)$.

Consider the homomorphism

$$\phi : R \rightarrow S$$

$$x_u \mapsto \begin{cases} s & u = b \\ t & u = c \\ \sigma_u s + \tau_u t & \text{else} \end{cases}.$$

Note that this maps homogeneous elements of degree (i, j) to elements of degree j . In a neighborhood of L_E , $\mathbf{F}_1(X_P)$ has coordinates σ_u, τ_u for $u \neq b, c$ and is cut out by the equations imposed by the condition that $\phi(I_P) = 0$. Let

$$J \subset K[\sigma_u, \tau_u \mid u \neq b, c]$$

denote the corresponding ideal. In other words, J is generated by the coefficients of elements of $f \in \phi(I_P)$ when f is viewed as a polynomial in s and t .

Set $v = (0, 1)$; note that $v \in P$, since it is in the convex hull of a, b, c .

Lemma 3.1. *Consider $u = (i, j) \in P \cap \mathbb{Z}^2$ with $j > 0$. Then modulo J , σ_u and τ_u can be written in terms of σ_v and τ_v . More precisely, we have*

$$\sigma_u = \begin{cases} 0 \mod J & i > 0 \\ \binom{j}{i+j} \sigma_v^{i+j} \tau_v^{-i} \mod J & i \leq 0 \end{cases}$$

$$\tau_u = \begin{cases} 0 \mod J & i > 1 \\ \binom{j}{i+j-1} \sigma_v^{i+j-1} \tau_v^{-i+1} \mod J & i \leq 1 \end{cases}.$$

Proof. Note that $u = ic + jv$ with $j \geq 1$, $i + j > 0$. This translates to the relations

$$\begin{aligned} x_u x_b^{i+j-1} - x_v^j x_c^i &\in I_P & i \geq 0; \\ x_u x_b^{i+j-1} x_c^{-i} - x_v^j &\in I_P & i \leq 0. \end{aligned}$$

By inspecting the images of these polynomials under ϕ , we arrive at the equations of the desired form. For example, the coefficient of s^{i+j} in $\phi(x_u x_b^{i+j-1} - x_v^j x_c^i)$ is $\sigma_u - \sigma_v^j$ if $i = 0$ and is σ_u if $i > 0$, leading respectively to the equations $\sigma_u = \sigma_v^j$ and $\sigma_u = 0$. \square

We now discuss various cases which can occur in our local study of $\mathbf{F}_1(X_P)$ around L_E , based on the value of μ_E . For each case, our task is twofold: firstly, determine some relations among the σ_u, τ_u ; secondly, show that they generate J .

3.1. Case $\mu_E > 2$. Fix a primitive edge E of P as above and suppose that $\mu_E \geq 2$. Then from above we have that $v = (1, 0)$, $w = (1, 1)$, and $(2, 1)$ are all in P . The relation

$$(2) \quad x_b x_w - x_c x_v = 0$$

implies that $\tau_v = \sigma_w = 0, \tau_w = \sigma_v$ modulo J , while the relation

$$x_b x_{(2,1)} - x_c x_w = 0$$

implies $\tau_w = 0$ modulo J . Hence, $\sigma_v = \tau_v = 0$ modulo J , and by Lemma 3.1 we may conclude that in $\mathbf{F}_1(X_P)$, the line L_E lies on a single irreducible component, which is isomorphic to ξ_1 .

3.2. Case $\mu_E = 2$. Fix a primitive edge E of P and suppose that $\mu_E = 2$. By using Equation (2), we get that modulo J , $\sigma_v = \tau_w$ and $\tau_v = \sigma_w = 0$.

Suppose that $\gamma(b) = \gamma(c) = \infty$. Then after possibly interchanging b and c , P is equal to some $P_{\alpha,\beta}$ with $\beta \geq 1$. By Lemma 3.1 and the above relations, we have that modulo J ,

$$\begin{aligned}\sigma_{(0,i)} &= \tau_{(1,i)} = \sigma_v^i \\ \tau_{(0,i)} &= \sigma_{(1,i)} = 0.\end{aligned}$$

However, these relations generate J , since the 2×2 minors of

$$\begin{pmatrix} \sigma_v^0 s & \sigma_v^1 s & \dots & \sigma_v^{\alpha-1} s & \sigma_v^0 t & \sigma_v^1 t & \dots & \sigma_v^{\beta-1} t \\ \sigma_v^1 s & \sigma_v^2 s & \dots & \sigma_v^\alpha s & \sigma_v^1 t & \sigma_v^2 t & \dots & \sigma_v^\beta t \end{pmatrix}$$

vanish, and I_P is generated by the 2×2 minors of (1). Hence in an affine neighborhood of the line L_E , $\mathbf{F}_1(X_P)$ is just the affine line \mathbb{A}^1 .

Suppose now that $\gamma(b) \neq \infty$. Let $a' = (-1, \gamma(b))$. The relation

$$(3) \quad x_{a'} x_b^{\gamma(b)-2} x_c - x_v^{\gamma(b)} = 0$$

imposes the condition $\sigma_v^{\gamma(b)} = 0$ modulo J . If $\gamma(c) \neq \infty$, we similarly obtain $\tau_w^{\gamma(c)} = 0$ modulo J . Coupled with the equations of Lemma 3.1, we obtain the following equations modulo J :

$$(4) \quad \sigma_{(u_1, u_2)} = \begin{cases} \sigma_v^{u_2} & u_1 = 0 \text{ and } u_2 < \gamma \\ 0 & \text{else} \end{cases}$$

$$(5) \quad \tau_{(u_1, u_2)} = \begin{cases} \sigma_v^{u_2} & u_1 = 1 \text{ and } u_2 < \gamma \\ 0 & \text{else} \end{cases}$$

$$(6) \quad \sigma_v^\gamma = 0.$$

where $\gamma = \min\{\gamma(b), \gamma(c)\}$.

We claim that these relations generate the ideal J . Indeed, the ideal $I_P \subset R$ is generated by homogeneous binomials $f = f_1 - f_2$. Consider such a binomial f of degree (i, j) . Then $\deg \phi(f) = j$, and if $j \geq \gamma$, then this must vanish modulo the relations of (4), (5), and (6). On the other hand, if $j < \gamma$, then the only coordinates x_u appearing in f must satisfy $u \in P_{\gamma-1, \gamma-1}$ and the argument used above for $P_{\alpha, \beta}$ shows that $\phi(f)$ vanishes modulo the above relations.

Thus, if not both $\gamma(b)$ and $\gamma(c)$ are infinite, we may conclude that L_E lies on a single irreducible component of $\mathbf{F}_1(X_P)$, which is isomorphic to ξ_γ .

3.3. Case $\mu_E = 1$. We first analyze the possible values of $\gamma(b)$ and $\gamma(c)$ under the assumption that $\mu_E = 1$. Without loss of generality, we assume that $\gamma(b) \geq \gamma(c)$. If $\gamma(b) = \infty$, then we have $P = P_{\alpha, 0}$ for some $\alpha \geq 1$. Note that $\gamma(c) = 2$ unless $\alpha = 1$, in which case X_P is \mathbb{P}^2 in its linear embedding and $\mathbf{F}_1(X_P)$ is simply $(\mathbb{P}^2)^* \cong \mathbb{P}^2$.

Suppose instead that $\gamma(b)$ is finite. Note that by definition of $\gamma(c)$, P must contain a lattice point of the form $(2 - \lambda\gamma(c), \gamma(c))$ for some $\lambda \in \mathbb{Z}$. But since $\mu_E = 1$, this implies that $\lambda \geq 1$. On the other hand, we must also have

$$\gamma(c)q \geq p(\lambda\gamma(c) - 2)$$

which in turn implies

$$\gamma(c) > (\gamma(b) - 1)(\gamma(c) - 2),$$

that is,

$$2(\gamma(b) + \gamma(c) - 1) > \gamma(b)\gamma(c).$$

This is only possible if $\gamma(c) = 2$ or $\gamma(b) = \gamma(c) = 3$.

We now consider the case $\gamma(c) = 2$. Then $d' = (0, 2) \in P$, and we have the relation

$$x_b x_{d'} - x_v^2 \in I_P.$$

This, together with Lemma 3.1, leads to the relations

$$(7) \quad \sigma_{(0,j)} = \sigma_v^j, \quad \tau_{(0,j)} = j\tau_v \sigma_v^{j-1} \quad (0, j) \in P$$

$$(8) \quad \tau_v^2 = 0$$

modulo J . Now, note that modulo the relation $\tau_v^2 = 0$, the 2×2 minors of the matrix

$$(9) \quad \begin{pmatrix} \sigma_v^0 s & \sigma_v^1 s + \tau_v t & \sigma_v^2 s + 2\sigma_v \tau_v t & \dots & \sigma_v^{\alpha-1} s + (\alpha-1)\sigma_v^{\alpha-2} \tau_v t \\ \sigma_v^1 s + \tau_v t & \sigma_v^2 s + 2\sigma_v \tau_v t & \sigma_v^3 s + 3\sigma_v^2 \tau_v t & \dots & \sigma_v^\alpha s + \alpha\sigma_v^{\alpha-1} \tau_v t \end{pmatrix}$$

all vanish. Hence, if $P = P_{\alpha,0}$ $\alpha \geq 2$, the relations of (7) and (8) generate J , and we conclude that in a neighborhood of L_E , $\mathbf{F}_1(X_P)$ is just $\mathbb{A}^1 \times \xi_2$.

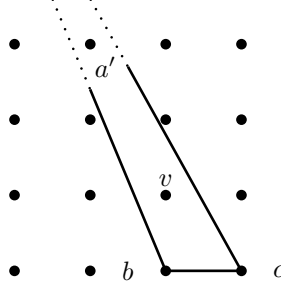
Suppose that $P \neq P_{\alpha,0}$ $\alpha \geq 2$; then $\gamma(b)$ is finite. We can then use Equation (3) to obtain the relations $\sigma_v^{\gamma(b)} = 0$ and $\sigma_{a'} = \gamma(b)\sigma_v^{\gamma(b)-1}\tau_v$ modulo J . With Lemma 3.1, this implies that $\sigma_u = \tau_u = 0$ modulo J for $u = (u_1, u_2)$ with $u_2 > \gamma(b)$.

We claim that these relations generate all of J . Indeed, consider a homogeneous binomial f of degree (i, j) in I_P . Arguing as in the case $\mu_E = 2$, if $\deg f > j$, then $\phi(f)$ will vanish modulo the above relations. Suppose instead that $\deg f \leq j$. If the only coordinates appearing in f are contained in some $P_{\alpha,0}$, then the vanishing of the minors of (9) shows that $\phi(f)$ must vanish modulo the relations. The only possible monomial which can appear in such f which is not in $P_{\alpha,0}$ is a' , and any f containing it must have the form $f = f_1 - f_2$, where $f_1 = x_{a'} x_b^{\gamma(b)} x_c$ and f_2 only involves coordinates in $P_{\alpha,0}$. But modulo equations in I_P only involving coordinates in $P_{\alpha,0}$, such a binomial is equal to the left hand side of (3), whose image under ϕ vanishes modulo the above relations.

Hence, these relations generate J , and we may conclude that the point L_E of $\mathbf{F}_1(X_P)$ lies on a single component, which is isomorphic to $\xi_{\gamma(b)} \times \xi_2$.

We now deal with the remaining case $\gamma(b) = \gamma(c) = 3$. Then P contains the lattice points $b = (0, 0)$, $c = (1, 0)$, $v = (0, 1)$, and $a' = (-1, 3)$, and these are the only lattice points $u = (u_1, u_2)$ contained in P with $u_2 \leq 3$, see Figure 3. Now, any relation among these four lattice points is a multiple of the cubic

$$x_v^3 - x_{a'} x_b x_c$$

FIGURE 3. The case $\mu_E = 1$, $\gamma(b) = \gamma(c) = 3$

which imposes exactly the conditions

$$\begin{aligned} \sigma_v^3 &= 0 & \tau_v^3 &= 0 \\ \sigma_{a'} &= 3\sigma_v^2\tau & \tau_{a'} &= 3\sigma\tau^2. \end{aligned}$$

Lemma 3.1 implies that for any $u \in P$ with $u_2 > 3$, we must have $\sigma_u = \tau_u = 0$. Now, arguments similar to those used in the previous cases imply that J is generated exactly by the above relations, so we may conclude that the point L_E of $\mathbf{F}_1(X_P)$ lies on a single component, which is isomorphic to $\xi_3 \times \xi_3$.

3.4. Conclusion of the proof. We now use our local study to completely describe $\mathbf{F}_1(X_P)$. A key point in our argument is that since $\mathbf{F}_1(X_P)$ is projective, any irreducible component of $\mathbf{F}_1(X_P)$ must contain a fixed point, that is, a point corresponding to one of the lines L_E . Thus, by the above local study, we already know the local structure of every component of our Fano scheme.

First, suppose that X_P is a rational normal scroll, that is, $P = P_{(\alpha, \beta)}$. As noted above, if $\alpha = 1, \beta = 0$, then X_P is just \mathbb{P}^2 , and its Fano scheme of lines is \mathbb{P}^2 as well. If $\alpha = \beta = 1$, P has four primitive edges E_1, \dots, E_4 , and each has $\mu_{E_i} = 2$ and the irreducible component containing L_{E_i} is locally just \mathbb{A}^1 . The only possibility for gluing four copies of \mathbb{A}^1 to get a projective scheme results in $\mathbb{P}^1 \sqcup \mathbb{P}^1$; in our case, this gluing can be seen quite explicitly. Alternatively, if $\alpha = \beta = 1$, then X_P is a nonsingular quadric in \mathbb{P}^3 , isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and it is well known that $\mathbf{F}_1(X_P) \cong \mathbb{P}^1 \sqcup \mathbb{P}^1$, the two irreducible components corresponding to the two different rulings coming from projecting to the first and second \mathbb{P}^1 factors.

If $\alpha \geq 2$ and $\beta = 0$, P has two primitive edges E_1, E_2 , for which we have $\mu_{E_1} = \mu_{E_2} = 1$. The local analysis shows that our Fano scheme is constructed by gluing two copies of $\mathbb{A}^1 \times \xi_2$, resulting in $\mathbf{F}_1(X_P) \cong \mathbb{P}^1 \times \xi_2$. If $\alpha \geq 2$ and $\beta = 1$, P has three primitive edges, one with $\mu = \alpha + 1$ and the other two with $\mu = 2$. For the edges with $\mu = 2$ we get two copies of \mathbb{A}^1 gluing to a \mathbb{P}^1 , and for the edge with $\mu = \alpha + 1 > 2$ we get an isolated point; hence $\mathbf{F}_1(X_P) \cong \mathbb{P}^1 \sqcup \xi_1$. If on the other hand, if $\alpha, \beta \geq 2$, then P has only two primitive edges, both with $\mu = 2$, and we get obtain two copies of \mathbb{A}^1 gluing to give $\mathbf{F}_1(X_P) \cong \mathbb{P}^1$.

If we assume instead that X_P is not a rational normal scroll, our local analysis shows that set theoretically, $\mathbf{F}_1(X_P)$ consists only of a finite number of fixed points, and the scheme structure is exactly as claimed in the theorem.

This concludes the proof of Theorem 2.2. \square

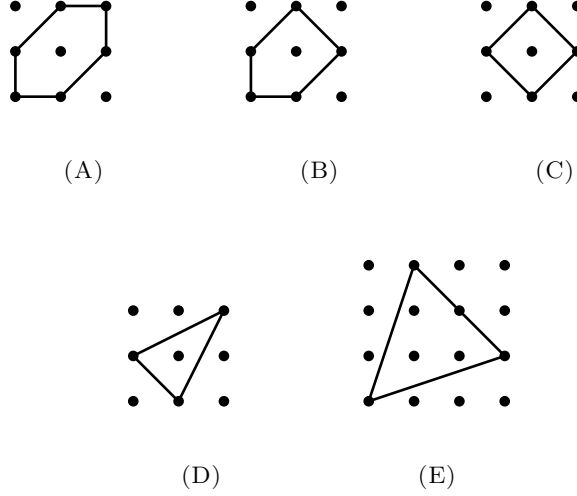


FIGURE 4. Examples for Theorem 2.2

4. APPLICATIONS AND EXAMPLES

We illustrate Theorem 2.2 with several examples.

Example 4.1. We consider the toric surfaces corresponding to the polygons pictured in Figure 4. For 4A, X_P is the smooth del Pezzo surface of degree 6 in its anticanonical embedding. Since P has exactly 6 primitive edges, we see that $\mathbf{F}_1(X_P)$ consists of six isolated points, and $\deg \mathbf{F}_1(X_P) = 6$.

For 4B, X_P is a singular del Pezzo surface of degree 5 in its anticanonical embedding. P has five primitive edges; for two we have $\mu_E = 3$ giving us two copies of ξ_1 , for two we have $\mu_E = 2$ with $\min\{\gamma(b), \gamma(c)\} = 2$ giving us two copies of ξ_2 , and the remaining edge has $\mu_E = 1$ with $\gamma(b) = \gamma(c) = 2$, giving us a copy of $\xi_2 \times \xi_2$. In total, we obtain $\deg \mathbf{F}_1(X_P) = 10$.

For 4C, X_P is a singular del Pezzo surface of degree 4 in its anticanonical embedding, a complete intersection of two singular quadrics. P has four primitive edges; each has $\mu_E = 1$ and $\gamma(b) = \gamma(c) = 2$, giving us four copies of $\xi_2 \times \xi_2$. In total, we obtain $\deg \mathbf{F}_1(X_P) = 16$.

For 4D, X_P is a singular del Pezzo surface of degree 3 in its anticanonical embedding, a singular cubic. P has three primitive edges; each has $\mu_E = 1$ and $\gamma(b) = \gamma(c) = 3$, giving us three copies of $\xi_3 \times \xi_3$. In total, we obtain $\deg \mathbf{F}_1(X_P) = 27$.

For 4E, X_P is a singular del Pezzo surface of degree 2 in the embedding given by twice the anticanonical class. P has two primitive edges; each has $\mu_E = 1$ and $\gamma(b), \gamma(c) = 2, 3$, giving us two copies of $\xi_2 \times \xi_3$. In total, we obtain $\deg \mathbf{F}_1(X_P) = 12$.

Given any projective scheme $X \subset \mathbb{P}^n$, its Fano scheme $\mathbf{F}_k(X)$ embeds as a subscheme of the Grassmannian $G(k+1, n+1)$. By $\deg \mathbf{F}_k(X)$, we mean the degree of $\mathbf{F}_k(X)$ obtained by composing the embedding $\mathbf{F}_k(X) \hookrightarrow G(k+1, n+1)$ with the Plücker embedding of $G(k+1, n+1)$. Note that if $\mathbf{F}_k(X)$ is zero-dimensional, the

number of k -planes contained in X is bounded above by $\deg \mathbf{F}_k(X)$, with equality if and only if $\mathbf{F}_k(X)$ is reduced.

Proposition 4.2. *Let S be a non-singular curve, $\mathcal{X} \subset \mathbb{P}^n \times S$ a flat projective family of K -schemes over S with general fiber Y and special fiber X . Suppose that $\dim \mathbf{F}_k(X) = \dim \mathbf{F}_k(Y)$. Then*

$$\deg \mathbf{F}_k(Y) \leq \deg \mathbf{F}_k(X).$$

Proof. The relative Fano scheme $\mathbf{F}_k(\mathcal{X}/S)$ gives a family over S whose fibers are the Fano schemes $\mathbf{F}_k(\mathcal{X}_s)$, where \mathcal{X}_s is the fiber of \mathcal{X} over $s \in S$. In general, this family will not be flat. By considering the special fiber of the closure of $\mathbf{F}_k(\mathcal{X}/S) \setminus \mathbf{F}_k(X)$, we get a subscheme $\mathbf{F}_k(X)' \subset \mathbf{F}_k(X)$ fitting into a flat family with $\mathbf{F}_k(Y)$ [EH00, Proposition II.29]. Since $\dim \mathbf{F}_k(X)' = \dim \mathbf{F}_k(Y) = \dim \mathbf{F}_k(X)$, we have $\deg \mathbf{F}_k(X)' \leq \deg \mathbf{F}_k(X)$. Finally, $\deg \mathbf{F}_k(Y) = \deg \mathbf{F}_k(X)'$ since flatness preserves Hilbert polynomials [Har77, Theorem 9.9]. \square

Using the above proposition coupled with Theorem 2.2, we may obtain upper bounds on the number of lines on certain non-toric surfaces. We illustrate this application by recovering bounds on the number of lines on certain del Pezzo surfaces.

Example 4.3. It is classically known that general del Pezzo surfaces of degrees 3, 4, and 5 in their anticanonical embeddings contain 27, 16, and 10 lines, respectively. Our methods reprove that these quantities are upper bounds for the number of lines. Indeed, such general surfaces have degenerations to the surfaces of 4D, 4C, and 4B, respectively, and the degrees of their Fano schemes calculated in Example 4.1 are exactly the quantities 27, 16, and 10.

Slightly more subtle arguments show that for any cubic surface $X \subset \mathbb{P}^3$ with $\dim \mathbf{F}_1(X) = 0$, we have $\deg \mathbf{F}_1(X) = 27$, cf. [EH00, Exercise IV.74]. Indeed, $\mathbf{F}_1(X)$ is a codimension-four local complete intersection in $G(2, 4)$, so for any degeneration as in Proposition 4.2, the relative Fano scheme $\mathbf{F}_k(\mathcal{X}/S)$ will be flat over S . Likewise, for any complete intersection X of 2 conics in \mathbb{P}^4 with $\dim \mathbf{F}_1(X) = 0$, we have $\deg \mathbf{F}_1(X) = 16$. Indeed, in this case $\mathbf{F}_1(X)$ is a codimension-six local complete intersection in $G(2, 5)$ and a similar argument applies.

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